HEAT CONDUCTIVITY OF A BODY WITH VARIABLE HEAT EXCHANGE COEFFICIENT

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The heat conduction problem for a half-space and a plate with a time dependent coefficient of heat exchange is solved. The solution is obtained by the combined utilization of an operational and a successive approximations method.

A special class of heat conduction problems with variable thermophysical coefficients, problems with a time-dependent heat exchange coefficient, exists. It is meaningful to examine these problems in the case when the heat exchange coefficient depends considerably more essentially on the time than on the temperature or coordinates. The process of heat exchange of a metal being rolled with beams and a surrounding medium is an example. During the pass the coefficient of heat exchange between the metal and the beams has the order of magnitude 5000 $W/m^2 \cdot deg$ [1]. As a rule, the time the metal is on the beams does not exceed 0.1 sec. The coefficient of heat exchange between the metal and the surrounding medium between passes has a magnitude of around 200 $W/m^2 \cdot deg$. The time between passes does not exceed several tens of seconds. It is conceivable that the dependence of the heat exchange coefficient on the temperature of the metal surface can be neglected during such short time intervals. The time dependence of the heat-exchange coefficient is considerably more essential.

Besides the technological reasons for the time change in the heat-exchange coefficient there is a number of others, namely: the change in the physical characteristics of the heat carrier (the velocity of motion, the degree of blackness, the density, etc.) or the time change in the state of the surface of the body being heated (oxidation, dust contamination, fissuring, etc.).

I. Semi-Infinite Body. Let us consider the temperature of the surrounding medium to be constant T_c. The heat-exchange coefficient on the boundary of the half-space is time dependent. Let us assume that this dependence can be approximated by the series

$$\alpha(t) = \sum_{n=1}^{\infty} \alpha_n \exp(-b_n t). \tag{1}$$

We obtain a function of the temperature field by solving the heat conduction differential equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial H^2} \tag{2}$$

under the boundary conditions

$$-\left.\frac{\partial v}{\partial H}\right|_{H=0} = \sum_{n=1}^{\infty} k_n \exp(-\beta_n \tau)\left[1 - (v)_{H=0}\right], \tag{3}$$

$$v\big|_{H=\infty} = v_\infty = \text{const}, \tag{4}$$

$$v\big|_{\tau=0} = v_\infty. \tag{5}$$

The system (2)-(5) is represented in dimensionless form.

Let us apply the Laplace – Carson integral transform to (2), whereupon we obtain

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$$\bar{v}(p,\ H) - \Theta_\infty = \bar{A}(p)\exp(-\sqrt{p}H). \tag{6}$$

Let us transform condition (3)

$$\bar{A}(p) + \sum_{n=1}^{\infty}\bar{A}(p+\beta_n)\frac{k_n\sqrt{p}}{p+\beta_n} = \sum_{n=1}^{\infty}\frac{k_n\sqrt{p}}{p+\beta_n}v'_\infty. \tag{7}$$

We solve the functional equation (7) by successive approximations. To do this, let us introduce the provisional parameter ξ which characterizes the degree of the approximation, and let us represent (7) as

$$\bar{A}(p) + \xi\sum_{n=1}^{\infty}\bar{A}(p+\beta_n)\frac{k_n\sqrt{p}}{p+\beta_n} = \sum_{n=1}^{\infty}\frac{k_n\sqrt{p}}{p+\beta_n}v'_\infty, \tag{8}$$

and let us expand the constant $\bar{A}(p)$ in powers of ξ

$$\bar{A}(p) = \bar{A}_0(p) + \xi\bar{A}_1(p) + \xi^2\bar{A}_2(p) + \ldots \tag{9}$$

Substituting (9) into (8) and equating coefficients of identical powers of ξ, we obtain the approximation for $\bar{A}(p)$. In particular, if we limit ourselves to the first member of the series from (7) we will have

$$\bar{A}_0(p) = v'_\infty\frac{\sqrt{p}}{p+\beta_1}, \tag{10}$$

$$\bar{A}_1(p) = -v'_\infty\frac{\sqrt{p}}{\sqrt{p+\beta_1}(p+2\beta_1)}, \tag{11}$$

$$\bar{A}_2(p) = v'_\infty\frac{\sqrt{p}}{\sqrt{(p+\beta_1)(p+2\beta_1)}(p+3\beta_1)}, \tag{12}$$

etc. For the k-th approximation

$$\bar{A}_k(p) = (-1)^k v'_\infty\frac{\sqrt{p}}{\sqrt{(p+\beta_1)(p+2\beta_1)\ldots(p+k\beta_1)}\,[p+(k+1)\beta_1]}. \tag{13}$$

Correspondingly we obtain the approximation to solve the formulated problem in the transform domain

$$[\bar{v}(p,\ H) - v_\infty]_0 \equiv \bar{v}_0(p,\ H), \tag{14}$$

$$[\bar{v}(p,\ H) - v_\infty]_1 \equiv \bar{v}_0(p,\ H) + \bar{v}_1(p,\ H), \tag{15}$$

$$[\bar{v}(p,\ H) - v_\infty]_2 \equiv \bar{v}_0(p,\ H) + \bar{v}_1(p,\ H) + \bar{v}_2(p,\ H), \tag{16}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[\bar{v}(p,\ H) - v_\infty]_k \equiv \bar{v}_0(p,\ H) + \bar{v}_1(p,\ H) + \bar{v}_2(p,\ H) + \ldots + \bar{v}_k(p,\ H), \tag{17}$$

where

$$\bar{v}_0(p,\ H) = \bar{A}_0(p)\exp(-\sqrt{p}H); \tag{18}$$

$$\bar{v}_1(p,\ H) = \bar{A}_1(p)\exp(-\sqrt{p}H); \tag{19}$$

$$\bar{v}_2(p,\ H) = \bar{A}_2(p)\exp(-\sqrt{p}H); \tag{20}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\bar{v}_k(p,\ H) = \bar{A}_k(p)\exp(-\sqrt{p}H). \tag{21}$$

Seeking the primitive of (18)-(20) we obtain an approximation for the function $v_k(\tau,\ H)$

$$v_0(\tau,\ H) = v'_\infty\int_0^\tau \exp(-\beta_1 t)\,\Phi_1(\tau - t)\,dt, \tag{22}$$

$$v_1(\tau,\ H) = -v'_\infty\int_0^\tau \exp(-2\beta_1 t)\,\frac{\operatorname{erf}(i\sqrt{\beta_1 t})}{i\sqrt{\beta_1}}\,\Phi_1(\tau - t)\,dt, \tag{23}$$

$$v_2(\tau,\ H) = v'_\infty\int_0^\tau \Phi_1(\tau - t)\,\Phi_2(t)\,dt \tag{24}$$

etc., where

$$\Phi_1(t) = \frac{\exp\left(-\frac{H^2}{4t}\right)}{\sqrt{\pi t}}; \tag{25}$$

$$\Phi_2(t) = \int_0^t \exp(-3\beta_1 y)\exp\left[-\frac{3}{2}\beta_1(t-y)\right] I_0\left[\frac{\beta_1}{2}(t-y)\right] dy. \tag{26}$$

We find the approximation of the desired temperature distribution function within the half-space from the following relationships:

$$[v(\tau, H) - v_\infty]_0 \equiv v_0(\tau, H) = \psi_0,$$

$$[v(\tau, H) - v_\infty]_1 \equiv v_0'(\tau, H) + v_1(\tau, H) = \psi_1,$$

$$[v(\tau, H) - v_\infty]_2 \equiv v_0(\tau, H) + v_1(\tau, H) + v_2(\tau, H) = \psi_2,$$

etc.

An analysis of (22)-(24) indicates the satisfactory convergence of the iteration. Presented in Figs. 1a and b are values of approximations of the temperature function ψ_k, computed as a function of the dimensionless time for the points H = 0.707 and H = 2 when $v_\infty = 0$, $v'_\infty = 1$, $\beta_1 = 1$. It is seen from the figures that the zero and first approximations "bracket" the second approximation, the first and second approximations "bracket" the third approximation, etc., which permits us to limit ourselves to the second approximation with sufficient accuracy for practical purposes.

It should be noted that there are no difficulties, in principle, in obtaining the solution of the heat conduction problem elucidated, keeping n terms of the series in the functional equation, although the final expressions for the approximations of the temperature function are more tedious to obtain than in the case considered when n = 1.

II. Infinite Plate. The temperature of the surrounding medium is T_C. The coefficient of heat exchange is defined by the relationship (1). We solve the problem of internal heat exchange for a plate by an analogous method to that expounded above, i.e., we solve the heat conduction differential equation

$$\frac{\partial v}{\partial \mathrm{Fo}} = \frac{\partial^2 \mathrm{Fo}}{\partial X^2} \tag{27}$$

under the boundary conditions

$$\left.\frac{\partial v}{\partial X}\right|_{X=1} = \mathrm{Bi}_1 \exp(-\delta_1 \mathrm{Fo})\left[1 - (v)_{X=1}\right], \tag{28}$$

$$\left.\frac{\partial v}{\partial X}\right|_{X=0} = 0, \tag{29}$$

$$v|_{\mathrm{Fo}=0} = v_0 = \mathrm{const} \tag{30}$$

by limiting ourselves to the first member of the series in (1).

Let us apply the Laplace – Carson transformation to (27). After using conditions (29) and (30) we obtain

$$\bar{v}(p, X) - v_0 = \bar{A}(p)\,\mathrm{ch}\sqrt{p}X. \tag{31}$$

The boundary condition (28) takes the form

$$\bar{A}(p) + \bar{A}(p+\delta_1)\frac{\mathrm{Bi}_1\sqrt{p}}{p+\delta_1}\,\frac{\mathrm{ch}\sqrt{p+\delta_1}}{\mathrm{sh}\sqrt{p}} = \frac{\mathrm{Bi}_1\sqrt{p}\;v_0'}{(p+\delta_1)\,\mathrm{sh}\sqrt{p}}. \tag{32}$$

We solve the functional equation (32) by the same method as for (7). Without presenting the intermediate operations, let us give the approximation for the solution of the problem in the transform domain

$$[\bar{v}(p, X) - v_0]_0 \equiv \bar{v}_0(p, X) = \bar{\varphi}_0, \tag{33}$$

$$[\bar{v}(p, X) - v_0]_1 \equiv \bar{v}_0(p, X) + \bar{v}_1(p, X) = \bar{\varphi}_1, \tag{34}$$

$$[\bar{v}(p, X) - v_0]_2 \equiv \bar{v}_0(p, X) + \bar{v}_1(p, X) + \bar{v}_2(p, X) = \bar{\varphi}_2, \tag{35}$$

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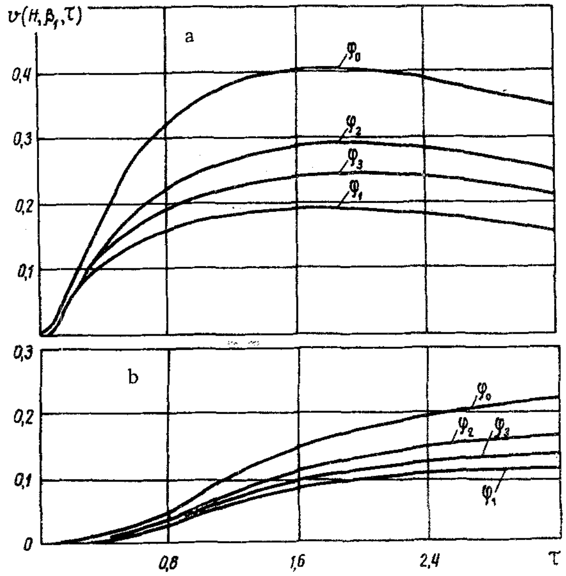

Fig. 1. Time variation of the approximations of the dimensionless temperature at the point H = 0.707 (a) and H = 2.0 (b) of a half-space.

$$[\bar{v}(p,\ X) - v_0]_k \equiv \sum_0^k \bar{v}_k(p,\ X) = \bar{\varphi}_k, \tag{36}$$

where

$$\bar{v}_0(p,\ X) = v_0' \frac{\mathrm{Bi}_1 \sqrt{p}\ \mathrm{ch}\sqrt{p}X}{(p+\delta_1)\ \mathrm{sh}\sqrt{p}}; \tag{37}$$

$$\bar{v}_1(p,\ X) = -v_0' \frac{\mathrm{Bi}_1^2 \sqrt{p}\ \mathrm{cth}\sqrt{p+\delta_1}\ \mathrm{ch}\sqrt{p}X}{\sqrt{p+\delta_1}\,(p+2\delta_1)\ \mathrm{sh}\sqrt{p}}; \tag{38}$$

$$\bar{v}_2(p,\ X) = v_0' \frac{\mathrm{Bi}_1^3 \sqrt{p}\ \mathrm{cth}\sqrt{p+\delta_1}\ \mathrm{cth}\sqrt{p+2\delta_1}\ \mathrm{ch}\sqrt{p}X}{\sqrt{(p+\delta_1)(p+2\delta_1)}\,(p+3\delta_1)\ \mathrm{sh}\sqrt{p}}. \tag{39}$$

Correspondingly, we have in the domain of the real variable

$$[v(\mathrm{Fo},\ X) - v_0]_k = \sum_0^k v_k(\mathrm{Fo},\ X) = \varphi_k,$$

where

$$v_0(\mathrm{Fo},\ X) = \mathrm{Bi}_1 v_0' f_1(X,\ \delta_1,\ \mathrm{Fo}), \tag{40}$$

$$v_1(\mathrm{Fo},\ X) = -\mathrm{Bi}_1^2 v_0' f_2(X,\ \delta_1,\ 2\delta_1,\ \mathrm{Fo}), \tag{41}$$

$$v_2(\mathrm{Fo},\ X) = \mathrm{Bi}_1^3 v_0' \int_0^{\mathrm{Fo}} \exp[-\delta_1(\mathrm{Fo}-t)]\, \Theta_0\left(\frac{1}{2},\ \mathrm{Fo}-t\right) f_2(X,\ 2\delta_1,\ 3\delta_1,\ t)\, dt, \tag{42}$$

and

$$f_1(X,\ \delta_1,\ \mathrm{Fo}) = \int_0^{\mathrm{Fo}} \exp[-\delta_1(\mathrm{Fo}-t)]\, \Theta_0\left(\frac{X}{2},\ t\right) dt;$$

$$f_2(X,\ \delta_1,\ 2\delta_1,\ \mathrm{Fo}) = \int_0^{\mathrm{Fo}} \exp[-\delta_1(\mathrm{Fo}-t)]\, \Theta_0\left(\frac{1}{2},\ \mathrm{Fo}-t\right) f_1(X,\ 2\delta_1,\ t)\, dt;$$

$\Theta_0(z,\ t)$ is the theta function defined by the relationship

$$\Theta_0(z,\ t) = 1 + 2\sum_{k=1}^{\infty} (-1)^k \exp(-\pi^2 k^2 t) \cos 2\pi k z. \tag{43}$$

It must be kept in mind that the series in the expression for the theta-function converges poorly for small values of the time. This circumstance demands the expenditure of a significant amount of labor in calculations for small Fo since in this case it is required to retain a large number of terms in the series in (43) to obtain a sufficiently accurate result. Hence, following [2], let us obtain a solution convenient for computations for small Fo. In particular, for the zero approximation let us transform (37) as follows:

$$\bar{v}_0(p,\ X)=\frac{v_0'\mathrm{Bi}_1\sqrt{p}\ \mathrm{ch}\sqrt{p}X}{(p+\delta_1)\,\mathrm{sh}\sqrt{p}}=\frac{v_0'\mathrm{Bi}_1\sqrt{p}}{p+\delta_1}\,\frac{\exp\left[-\sqrt{p}(1-X)\right]+\exp\left[-\sqrt{p}(1+X)\right]}{1-\exp\left(-2\sqrt{p}\right)}$$

$$=\frac{v_0'\mathrm{Bi}_1\sqrt{p}}{p+\delta_1}\sum_{n=0}^{\infty}\left\{\exp\left[-\sqrt{p}(1-X+2n)\right]+\exp\left[-\sqrt{p}(1+X+2n)\right]\right\}. \tag{44}$$

The inversion of (44) causes no difficulties

$$v_0(\mathrm{Fo},\ X)=\mathrm{Bi}_1 v_0' f_3(X,\ \delta_1,\ \mathrm{Fo}), \tag{45}$$

where

$$f_3(X,\ \delta_1,\ \mathrm{Fo})=\frac{1}{\delta_1}\int_0^{\mathrm{Fo}}\left\{1-\exp\left[-\delta_1(\mathrm{Fo}-t)\right]\right\}\frac{1}{2t\sqrt{\pi t}}$$

$$\times\sum_{n=0}^{\infty}\left\{\exp\left[-\frac{(1-X+2n)^2}{4t}\right]+\exp\left[-\frac{(1+X+2n)^2}{4t}\right]\right\}dt.$$

Analogously transforming (38) and (39), and also keeping in mind that large values of the parameter p correspond to small values of the time, and the coth z is practically unity for $z > 2$, the successive approximations for the temperature function can be obtained in a form convenient for calculations with small Fo:

$$v_1(\mathrm{Fo},\ X)=-\mathrm{Bi}_1^2 v_0' f_4(X,\ \delta_1,\ 2\delta_1,\ \mathrm{Fo}),$$

$$v_2(\mathrm{Fo},\ X)=\mathrm{Bi}_1^3 v_0'\int_0^{\mathrm{Fo}}\frac{\exp\left[-\delta_1(\mathrm{Fo}-t)\right]}{\sqrt{\pi(\mathrm{Fo}-t)}}f_4(X,\ 2\delta_1,\ 3\delta_1,\ t)\,dt, \tag{46}$$

where

$$f_4(X,\ \delta_1,\ 2\delta_1,\ \mathrm{Fo})=\int_0^{\mathrm{Fo}}\frac{\exp\left[-\delta_1(\mathrm{Fo}-t)\right]}{\sqrt{\pi(\mathrm{Fo}-t)}}f_3(X,\ 2\delta_1,\ t)\,dt.$$

It should also be noted that the approximations considered for the temperature function of a plate turn out to converge well only for values of the criterion $\mathrm{Bi}_1 \leq 1$. When $\mathrm{Bi}_1 > 1$ the iteration turns out to be poorly convergent, where the convergence is worse the larger the number Bi_1. Hence, it is interesting to obtain a solution which would converge well in the range of values of $\mathrm{Bi}_1 > 1$. To do this let us utilize the following thermal similarity criterion:

$$H=h_1x\,(0\leqslant H\leqslant \mathrm{Bi}_1),\quad \tau=ah_1^2t.$$

The solution of the heat conduction problem for a plate then reduces to solving the differential equation (2) under the boundary conditions

$$\left.\frac{\partial v}{\partial H}\right|_{H=\mathrm{Bi}_1}=\exp(-\beta_1\tau)\left[1-(v)_{H=\mathrm{Bi}_1}\right], \tag{47}$$

$$\left.\frac{\partial v}{\partial H}\right|_{H=0}=0, \tag{48}$$

$$v|_{\tau=0}=v_0=\mathrm{const}. \tag{49}$$

The Laplace – Carson transformation converts the boundary condition (40) into the following functional equation

$$\bar{A}(p)+\bar{A}(p+\beta_1)\frac{\sqrt{p}}{p+\beta_1}\,\frac{\mathrm{ch}\left(\sqrt{p+\beta_1}\,\mathrm{Bi}_1\right)}{\mathrm{sh}\left(\sqrt{p}\,\mathrm{Bi}_1\right)}=\frac{v_0'P}{p+\beta_1}. \tag{50}$$

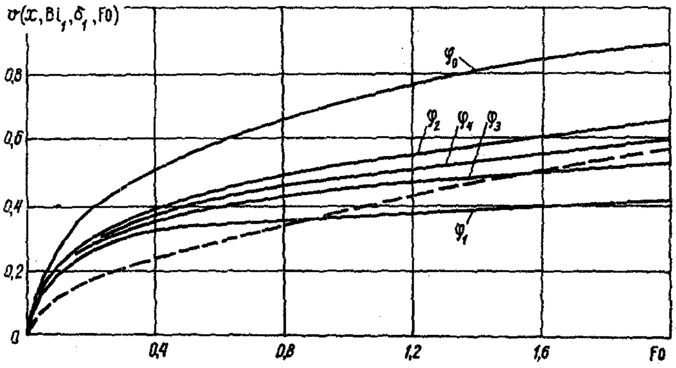

Fig. 2. Time variation of the approximations of the dimensionless temperature at the point X = 1 of a plate.

The method of solving the functional equation (50) and all the problems is analogous to that examined above. Hence, without performing the intermediate operations (they are obvious), let us present the final expressions for the functions $v_k(\tau, H)$

$$v_0(\tau, H) = \frac{v_0'}{Bi_1} \int_0^{\tau} \exp\left[-\beta_1(\tau - t)\right] \Theta_0\left(\frac{H}{2Bi_1}, \frac{t}{Bi_1^2}\right) dt, \tag{51}$$

$$v_1(\tau, H) = -\frac{v_0'}{Bi_1^2} \int_0^{\tau} \exp\left[-\beta_1(\tau - t)\right] \Theta_0\left(\frac{1}{2}, \frac{\tau - t}{Bi_1^2}\right) f_5(H, 2\beta_1, t)\, dt, \tag{52}$$

$$v_2(\tau, H) = \frac{v_0'}{Bi_1^3} \int_0^{\tau} \exp\left[-\beta_1(\tau - t)\right] \Theta_0\left(\frac{1}{2}, \frac{\tau - t}{Bi_1^2}\right) f_6(H, 2\beta_1, 3\beta_1, t)\, dt, \tag{53}$$

where

$$f_5(H, 2\beta_1, t) = \int_0^{t} \exp\left[-2\beta_1(t - \varphi)\right] \Theta_0\left(\frac{H}{2Bi_1}, \frac{\varphi}{Bi_1^2}\right) d\varphi; \tag{54}$$

$$f_6(H, 2\beta_1, 3\beta_1, t) = \int_0^{t} \exp\left[-2\beta_1(t - \varphi)\right] \times \Theta_0\left(\frac{1}{2}, \frac{t - \varphi}{Bi_1^2}\right) f_5(H, 3\beta_1, \varphi)\, d\varphi. \tag{55}$$

Approximations of the temperature function on a plate surface (Fig. 2) have been computed for the following values of the arguments $Bi_1 = 1.0$; $X = 1.0$; $\delta_1 = 1.0$; $v_0' = 1$ on the basis of (40)-(42).

The results of the computation show that for practical purposes (as in the case of the half-space), it is possible to limit oneself to the second approximation. The dashed line has been computed for the case of an invariant heat-exchange coefficient (the average during the period under consideration). The mean value of the Biot criterion has been defined by the method customary in this case

$$Bi_{av} = \frac{1}{Fo} \int_0^{Fo} \exp(-\delta_1 t)\, dt.$$

The computed mean value of the Biot criterion in the range $0 \le Fo \le 2$ is 0.432 (for $\delta_1 = 1.0$).

A comparison of the results of a computation by the theory elucidated and by a method utilizing the mean value of the Biot criterion during the period studied shows the possibility of significant error appearing in this latter case.

In conclusion, let us note yet another quite important practical case of the time variation in the heat-exchange coefficient according to the harmonic law

$$\alpha(t) = \frac{C_0}{2} + \sum_{m=1}^{\infty} (a_m \cos \omega_m t + b_m \sin \omega_m t),$$

which will be examined later.

NOTATION

T	is the body temperature;
T_c	is the temperature of the heat liberating medium;
$v = T/T_c$;	
$\alpha(t)$	are the heat exchange coefficients;
t	is the time;
$\tau = ah_1^2t$; a	are the coefficients of temperature conductivity;
h_1	is the relative coefficient of heat exchange;
$H = h_1x$	is the relative thermal resistivity of a layer of thickness $\bar{\bar{X}}$;
$k_n = \alpha/\alpha_1$; p	are the complex parameters;
$\beta_n = b/ah_n^2$;	
$v_\infty^1 = 1 - v_\infty$;	
$I_0(z)$	are the Bessel functions of imaginary argument of the first kind;
$Bi_1 = h_1R$	is the Biot criterion;
$2R$	is the plate thickness;
$\delta_1 = b_1R^2/a$;	
$Fo = at/R^2$	are the Fourier criteria;
$X = x/R$	is the relative coordinate;
$v_0^1 = 1 - v_0$.	

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